

# Feynman Diagrams for Beginners

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## Contents

<b>1</b>	<b>Natural units</b>	<b>2</b>
<b>2</b>	<b>Single-particle Dirac equation</b>	<b>3</b>
2.1	The Dirac equation . . . . .	3
2.2	The adjoint Dirac equation and the Dirac current . . . . .	6
2.3	Free-particle solutions of the Dirac equation . . . . .	6
<b>3</b>	<b>Free quantum fields</b>	<b>8</b>
3.1	Spin 0: scalar field . . . . .	10
3.2	Spin 1/2: the Dirac field . . . . .	10
3.3	Spin 1: vector field . . . . .	10
<b>4</b>	<b>Golden rules for decays and scatterings</b>	<b>11</b>
<b>5</b>	<b>Feynman diagrams</b>	<b>13</b>
<b>6</b>	<b><math>e^+e^- \rightarrow \mu^+\mu^-</math> in QED</b>	<b>16</b>
6.1	Summing over polarizations . . . . .	16
6.2	Casimir trick . . . . .	17
6.3	Traces and contraction identities of $\gamma$ matrices . . . . .	17
6.4	Kinematics in the center-of-mass frame . . . . .	19
6.5	Integration over two-particle phase space . . . . .	20
6.6	Summary of steps . . . . .	21
6.7	Mandelstam variables . . . . .	21
	<b>Appendix: Doing Feynman diagrams on a computer</b>	<b>23</b>

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## 1 Natural units

To describe kinematics of some physical system we are free to choose units of measure of the three basic kinematical physical quantities: *length* ( $L$ ), *mass* ( $M$ ) and *time* ( $T$ ). Equivalently, we may choose any three linearly independent combinations of these quantities. The choice of  $L$ ,  $T$  and  $M$  is usually made (e.g. in SI system of units) because they are most convenient for description of our immediate experience. However, elementary particles experience a different world, one governed by the laws of relativistic quantum mechanics.

Natural units in relativistic quantum mechanics are chosen in such a way that fundamental constants of this theory,  $c$  and  $\hbar$ , are both equal to one.

$[c] = LT^{-1}$ ,  $[\hbar] = ML^{-2}T^{-1}$ , and to completely fix our system of units we specify the unit of energy ( $ML^2T^{-2}$ ):

$$1 \text{ GeV} = 1.6 \cdot 10^{-10} \text{ kg m}^2 \text{ s}^{-2} \quad \approx m_p, m_n .$$

What we do in practice is:

- we ignore  $\hbar$  and  $c$  in formulae and only restore them at the end (if at all)
- we measure *everything* in GeV,  $\text{GeV}^{-1}$ ,  $\text{GeV}^2$ , ...

### Example: Thomson cross section

Total cross section for scattering of classical electromagnetic radiation by a free electron (Thomson scattering) is, in natural units,

$$\sigma_{\text{T}} = \frac{8\pi\alpha^2}{3m_e^2} . \quad (1)$$

To restore  $\hbar$  and  $c$  we insert them in the above equation with general powers  $\alpha$  and  $\beta$ , which we determine by requiring that cross section has the dimension of area ( $L^2$ ):

$$\sigma_{\text{T}} = \frac{8\pi\alpha^2}{3m_e^2} \hbar^\alpha c^\beta \quad (2)$$

$$[\sigma] = L^2 = \frac{1}{M^2} (ML^2T^{-1})^\alpha (LT^{-1})^\beta$$

$$\Rightarrow \alpha = 2, \quad \beta = -2,$$

i.e.

$$\sigma_{\text{T}} = \frac{8\pi\alpha^2 \hbar^2}{3m_e^2 c^2} = 0.665 \cdot 10^{-24} \text{ cm}^2 = 665 \text{ mb} . \quad (3)$$

Linear independence of  $\hbar$  and  $c$  implies that this can always be done in a unique way. ■

Following conversion relations are often useful:

$$\begin{aligned}
1 \text{ fermi} &= 5.07 \text{ GeV}^{-1} \\
1 \text{ GeV}^{-2} &= 0.389 \text{ mb} \\
1 \text{ GeV}^{-1} &= 6.582 \cdot 10^{-25} \text{ s} \\
1 \text{ kg} &= 5.61 \cdot 10^{26} \text{ GeV} \\
1 \text{ m} &= 5.07 \cdot 10^{15} \text{ GeV}^{-1} \\
1 \text{ s} &= 1.52 \cdot 10^{24} \text{ GeV}^{-1}
\end{aligned}$$

**Exercise 1** Check these relations.

Calculating with GeVs is much more elegant. Using  $m_e = 0.511 \cdot 10^{-3} \text{ GeV}$  we get

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 1709 \text{ GeV}^{-2} = 665 \text{ mb} . \quad (4)$$

right away.

**Exercise 2** The decay width of the  $\pi^0$  particle is

$$\Gamma = \frac{1}{\tau} = 7.7 \text{ eV} . \quad (5)$$

Calculate its lifetime  $\tau$  in seconds. (By the way, particle's half-life is equal to  $\tau \ln 2$ .)

## 2 Single-particle Dirac equation

### 2.1 The Dirac equation

Turning the relativistic energy equation

$$E^2 = \mathbf{p}^2 + m^2 . \quad (6)$$

into a differential equation using the usual substitutions

$$\mathbf{p} \rightarrow -i\nabla , \quad E \rightarrow i\frac{\partial}{\partial t} , \quad (7)$$

results in the Klein-Gordon equation:

$$(\square + m^2)\psi(x) = 0 , \quad (8)$$

which, interpreted as a single-particle wave equation, has problematic negative energy solutions. This is due to the negative root in  $E = \pm\sqrt{\mathbf{p}^2 + m^2}$ . Namely, in relativistic *mechanics* this negative root could be ignored, but here one must keep *all* of the complete set of solutions to a differential equation.

In order to overcome this problem Dirac tried the ansatz<sup>†</sup>

$$(i\beta^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0 \quad (9)$$

with  $\beta^\mu$  and  $\gamma^\nu$  to be determined by requiring consistency with the Klein-Gordon equation. This requires  $\gamma^\mu = \beta^\mu$  and

$$\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = \partial^\mu\partial_\mu, \quad (10)$$

which in turn implies

$$\begin{aligned} (\gamma^0)^2 &= 1, & (\gamma^i)^2 &= -1, \\ \{\gamma^\mu, \gamma^\nu\} &\equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 & \text{for } \mu \neq \nu. \end{aligned}$$

This can be compactly written in form of the *anticommutation relations*

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (11)$$

These conditions are obviously impossible to satisfy with  $\gamma$ 's being equal to usual numbers, but we can satisfy them by taking  $\gamma$ 's equal to (at least) four-by-four matrices.

Now, to satisfy (9) it is enough that one of the two factors in that equation is zero, and by convention we require this from the second one. Thus we obtain the *Dirac equation*:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \quad (12)$$

$\psi(x)$  now has four components and is called the *Dirac spinor*.

One of the most frequently used representations for  $\gamma$  matrices is the original Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (13)$$

where  $\sigma^i$  are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$

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<sup>†</sup>ansatz: guess, trial solution (from German *Ansatz*: start, beginning, onset, attack)

This representation is very convenient for the non-relativistic approximation, since then the dominant energy terms  $(i\gamma^0\partial_0 - \dots - m)\psi(0)$  turn out to be diagonal.

Two other often used representations are

- the Weyl (or chiral) representation — convenient in the ultra-relativistic regime (where  $E \gg m$ )
- the Majorana representation — makes the Dirac equation real; convenient for *Majorana fermions* for which antiparticles are equal to particles

(*Question:* Why can we choose at most one  $\gamma$  matrix to be diagonal?)

Properties of the Pauli matrices:

$$\sigma^{i\dagger} = \sigma^i \quad (15)$$

$$\sigma^{i*} = (i\sigma^2)\sigma^i(i\sigma^2) \quad (16)$$

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \quad (17)$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \quad (18)$$

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \quad (19)$$

where  $\epsilon^{ijk}$  is the totally antisymmetric Levi-Civita tensor ( $\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1$ ,  $\epsilon^{213} = \epsilon^{321} = \epsilon^{132} = -1$ , and all other components are zero).

**Exercise 3** Prove that  $(\boldsymbol{\sigma} \cdot \mathbf{a})^2 = \mathbf{a}^2$  for any three-vector  $\mathbf{a}$ .

**Exercise 4** Using properties of the Pauli matrices, prove that  $\gamma$  matrices in the Dirac representation satisfy  $\{\gamma^i, \gamma^j\} = 2g^{ij} = -2\delta^{ij}$ , in accordance with the anticommutation relations. (Other components of the anticommutation relations,  $(\gamma^0)^2 = 1$ ,  $\{\gamma^0, \gamma^i\} = 0$ , are trivial to prove.)

**Exercise 5** Show that in the Dirac representation  $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$ .

**Exercise 6** Determine the Dirac Hamiltonian by writing the Dirac equation in the form  $i\partial\psi/\partial t = H\psi$ . Show that the hermiticity of the Dirac Hamiltonian implies that the relation from the previous exercise is valid regardless of the representation.

The *Feynman slash* notation,  $\not{a} \equiv a_\mu\gamma^\mu$ , is often used.

## 2.2 The adjoint Dirac equation and the Dirac current

For constructing the Dirac current we need the equation for  $\psi(x)^\dagger$ . By taking the Hermitian adjoint of the Dirac equation we get

$$\psi^\dagger \gamma^0 (i \overleftarrow{\not{\partial}} + m) = 0 ,$$

and we define the *adjoint* spinor  $\bar{\psi} \equiv \psi^\dagger \gamma^0$  to get the *adjoint Dirac equation*

$$\bar{\psi}(x) (i \overleftarrow{\not{\partial}} + m) = 0 .$$

$\bar{\psi}$  is introduced not only to get aesthetically pleasing equations but also because it can be shown that, unlike  $\psi^\dagger$ , it transforms covariantly under the Lorentz transformations.

**Exercise 7** Check that the current  $j^\mu = \bar{\psi} \gamma^\mu \psi$  is conserved, i.e. that it satisfies the continuity relation  $\partial_\mu j^\mu = 0$ .

Components of this relativistic four-current are  $j^\mu = (\rho, \mathbf{j})$ . Note that  $\rho = j^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi > 0$ , i.e. that probability is positive definite.

## 2.3 Free-particle solutions of the Dirac equation

Since we are preparing ourselves for the perturbation theory calculations, we need to consider only free-particle solutions. For solutions in various potentials, see the literature.

The fact that Dirac spinors satisfy the Klein-Gordon equation suggests the ansatz

$$\psi(x) = u(\mathbf{p}) e^{-ipx} , \quad (20)$$

which after inclusion in the Dirac equation gives the *momentum space Dirac equation*

$$(\not{p} - m)u(\mathbf{p}) = 0 . \quad (21)$$

This has two positive-energy solutions

$$u(\mathbf{p}, \sigma) = N \begin{pmatrix} \chi^{(\sigma)} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \chi^{(\sigma)} \end{pmatrix} , \quad \sigma = 1, 2 , \quad (22)$$

where

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad (23)$$

and two negative-energy solutions which are then interpreted as positive-energy *antiparticle* solutions

$$v(\mathbf{p}, \sigma) = -N \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} (i\sigma^2)\chi^{(\sigma)} \\ (i\sigma^2)\chi^{(\sigma)} \end{pmatrix}, \quad \sigma = 1, 2, \quad E > 0. \quad (24)$$

$N$  is the normalization constant to be determined later. The momentum-space Dirac equation for antiparticle solutions is

$$(\not{p} + m)v(\mathbf{p}, \sigma) = 0. \quad (25)$$

It can be shown that the two solutions, one with  $\sigma = 1$  and another with  $\sigma = 2$ , correspond to the two spin states of the spin-1/2 particle.

**Exercise 8** Determine momentum-space Dirac equations for  $\bar{u}(\mathbf{p}, \sigma)$  and  $\bar{v}(\mathbf{p}, \sigma)$ .

### Normalization

In non-relativistic single-particle quantum mechanics normalization of a wavefunction is straightforward. Probability that the particle is somewhere in space is equal to one, and this translates into the normalization condition  $\int \psi^* \psi dV = 1$ . On the other hand, we will eventually use spinors (22) and (24) in many-particle quantum field theory so their normalization is not unique. We will choose normalization convention where we have  $2E$  particles in the unit volume:

$$\int_{\text{unit volume}} \rho dV = \int_{\text{unit volume}} \psi^\dagger \psi dV = 2E \quad (26)$$

This choice is relativistically covariant because the Lorentz contraction of the volume element is compensated by the energy change. There are other normalization conventions with other advantages.

**Exercise 9** Determine the normalization constant  $N$  conforming to this choice.

### Completeness

**Exercise 10** Using the explicit expressions (22) and (24) show that

$$\sum_{\sigma=1,2} u(\mathbf{p}, \sigma)\bar{u}(\mathbf{p}, \sigma) = \not{p} + m, \quad (27)$$

$$\sum_{\sigma=1,2} v(\mathbf{p}, \sigma)\bar{v}(\mathbf{p}, \sigma) = \not{p} - m. \quad (28)$$

These relations are often needed in calculations of Feynman diagrams with unpolarized fermions. See later sections.

### Parity and bilinear covariants

The parity transformation:

- $P : \mathbf{x} \rightarrow -\mathbf{x}, t \rightarrow t$
- $P : \psi \rightarrow \gamma^0 \psi$

**Exercise 11** Check that the current  $j^\mu = \bar{\psi} \gamma^\mu \psi$  transforms as a vector under parity i.e. that  $j^0 \rightarrow j^0$  and  $\mathbf{j} \rightarrow -\mathbf{j}$ .

Any fermion current will be of the form  $\bar{\psi} \Gamma \psi$ , where  $\Gamma$  is some four-by-four matrix. For construction of interaction Lagrangian we want to use only those currents that have definite Lorentz transformation properties. To this end we first define two new matrices:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \stackrel{\text{Dirac rep.}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \{\gamma^5, \gamma^\mu\} = 0, \quad (29)$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \sigma^{\mu\nu} = -\sigma^{\nu\mu}. \quad (30)$$

Now  $\bar{\psi} \Gamma \psi$  will transform covariantly if  $\Gamma$  is one of the matrices given in the following table. Transformation properties of  $\bar{\psi} \Gamma \psi$ , the number of different  $\gamma$  matrices in  $\Gamma$ , and the number of components of  $\Gamma$  are also displayed.

$\Gamma$	transforms as	# of $\gamma$ 's	# of components
1	scalar	0	1
$\gamma^\mu$	vector	1	4
$\sigma^{\mu\nu}$	tensor	2	6
$\gamma^5 \gamma^\mu$	axial vector	3	4
$\gamma^5$	pseudoscalar	4	1

This exhausts all possibilities. The total number of components is 16, meaning that the set  $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5 \gamma^\mu, \gamma^5\}$  makes a complete basis for any four-by-four matrix. Such  $\bar{\psi} \Gamma \psi$  currents are called *bilinear covariants*.

## 3 Free quantum fields

Single-particle Dirac equation is (a) not exactly right even for single-particle systems such as the H-atom, and (b) unable to treat many-particle processes such as the  $\beta$ -decay  $n \rightarrow p e^- \bar{\nu}$ . We have to upgrade to quantum field theory.



Any Dirac field is some superposition of the complete set

$$u(\mathbf{p}, \sigma)e^{-ipx}, \quad v(\mathbf{p}, \sigma)e^{ipx}, \quad \sigma = 1, 2, \quad \mathbf{p} \in \mathbb{R}^3$$

and we can write it as

$$\psi(x) = \sum_{\sigma} \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} [u(\mathbf{p}, \sigma)a(\mathbf{p}, \sigma)e^{-ipx} + v(\mathbf{p}, \sigma)a^{c\dagger}(\mathbf{p}, \sigma)e^{ipx}] . \quad (31)$$

Here  $1/\sqrt{(2\pi)^3 2E}$  is a normalization factor (there are many different conventions), and  $a(\mathbf{p}, \sigma)$  and  $a^{c\dagger}(\mathbf{p}, \sigma)$  are expansion coefficients. To make this a *quantum Dirac field* we promote these coefficients to the rank of operators by imposing the *anticommutation* relations

$$\{a(\mathbf{p}, \sigma), a^{\dagger}(\mathbf{p}', \sigma')\} = \delta_{\sigma\sigma'}\delta^3(\mathbf{p} - \mathbf{p}'), \quad (32)$$

and similarly for  $a^c(\mathbf{p}, \sigma)$ . (For bosonic fields we would have a *commutation* relations instead.) This is similar to the promotion of position and momentum to the rank of operators by the  $[x_i, p_j] = i\hbar\delta_{ij}$  commutation relations, which is why is this transition from the single-particle quantum theory to the quantum field theory sometimes called *second quantization*.

Operator  $a^{\dagger}$ , when operating on vacuum state  $|0\rangle$ , creates one-particle state  $|\mathbf{p}, \sigma\rangle$

$$a^{\dagger}(\mathbf{p}, \sigma)|0\rangle = |\mathbf{p}, \sigma\rangle, \quad (33)$$

and this is the reason that it is named a *creation* operator. Similarly,  $a$  is an *annihilation* operator

$$a(\mathbf{p}, \sigma)|\mathbf{p}, \sigma\rangle = |0\rangle, \quad (34)$$

and  $a^{c\dagger}$  and  $a^c$  are creation and annihilation operators for antiparticle states ( $c$  in  $a^c$  stands for “conjugated”).

Processes in particle physics are mostly calculated in the framework of the theory of such fields — *quantum field theory*. This theory can be described at various levels of rigor but in any case is complicated enough to be beyond the scope of these notes.

However, predictions of quantum field theory pertaining to the elementary particle interactions can often be calculated using a relatively simple “recipe” — *Feynman diagrams*.

Before we turn to describing the method of Feynman diagrams, let us just specify other quantum fields that take part in the elementary particle physics interactions. All these are *free* fields, and interactions are treated as their perturbations. Each particle type (electron, photon, Higgs boson, ...) has its own quantum field.

### 3.1 Spin 0: scalar field

- E.g. Higgs boson, pions, ...

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} [a(\mathbf{p})e^{-ipx} + a^\dagger(\mathbf{p})e^{ipx}] \quad (35)$$

### 3.2 Spin 1/2: the Dirac field

- E.g. quarks, leptons

We have already specified the Dirac spin-1/2 field. There are other types: Weyl and Majorana spin-1/2 fields but they are beyond our scope.

### 3.3 Spin 1: vector field

Either

- massive (e.g. W,Z weak bosons) or
- massless (e.g. photon)

$$A^\mu(x) = \sum_\lambda \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} [\epsilon^\mu(\mathbf{p}, \lambda)a(\mathbf{p}, \lambda)e^{-ipx} + \epsilon^{\mu*}(\mathbf{p}, \lambda)a^\dagger(\mathbf{p}, \lambda)e^{ipx}] \quad (36)$$

$\epsilon^\mu(\mathbf{p}, \lambda)$  is a polarization vector. For massive particles it obeys

$$p_\mu \epsilon^\mu(\mathbf{p}, \lambda) = 0 \quad (37)$$

automatically, whereas in the massless case this condition can be imposed thanks to gauge invariance (Lorentz gauge condition). This means that there are only three independent polarizations of a massive vector particle:  $\lambda = 1, 2, 3$  or  $\lambda = +, -, 0$ . In massless case gauge symmetry can be further exploited to eliminate one more polarization state leaving us with only two:  $\lambda = 1, 2$  or  $\lambda = +, -$ .

Normalization of polarization vectors is such that

$$\epsilon^*(\mathbf{p}, \lambda) \cdot \epsilon(\mathbf{p}, \lambda) = -1. \quad (38)$$

E.g. for a massive particle moving along the  $z$ -axis ( $p = (E, 0, 0, |\mathbf{p}|)$ ) we can take

$$\epsilon(\mathbf{p}, \pm) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}, \quad \epsilon(\mathbf{p}, 0) = \frac{1}{m} \begin{pmatrix} |\mathbf{p}| \\ 0 \\ 0 \\ E \end{pmatrix} \quad (39)$$

**Exercise 12** Calculate

$$\sum_{\lambda} \epsilon^{\mu*}(\mathbf{p}, \lambda) \epsilon^{\nu}(\mathbf{p}, \lambda)$$

Hint: Write it in the most general form  $(Ag^{\mu\nu} + Bp^{\mu}p^{\nu})$  and then determine  $A$  and  $B$ .

The obtained result obviously cannot be simply extrapolated to the massless case via the limit  $m \rightarrow 0$ . Gauge symmetry makes massless polarization sum somewhat more complicated but for the purpose of the Feynman diagram calculations it is permissible to use just the following relation

$$\sum_{\lambda} \epsilon^{\mu*}(\mathbf{p}, \lambda) \epsilon^{\nu}(\mathbf{p}, \lambda) = -g^{\mu\nu} .$$

## 4 Golden rules for decays and scatterings

Principal experimental observables of particle physics are

- scattering cross section  $\sigma(1 + 2 \rightarrow 1' + 2' + \dots + n')$
- decay width  $\Gamma(1 \rightarrow 1' + 2' + \dots + n')$

On the other hand, theory is defined in terms of Lagrangian density of quantum fields, e.g.

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 .$$

How to calculate  $\sigma$ 's and  $\Gamma$ 's from  $\mathcal{L}$ ?

To calculate rate of transition from the state  $|\alpha\rangle$  to the state  $|\beta\rangle$  in the presence of the interaction potential  $V_I$  in non-relativistic quantum theory we have the Fermi's Golden Rule

$$\left( \begin{array}{c} \alpha \rightarrow \beta \\ \text{transition rate} \end{array} \right) = \frac{2\pi}{\hbar} |\langle \beta | V_I | \alpha \rangle|^2 \times \left( \begin{array}{c} \text{density of final} \\ \text{quantum states} \end{array} \right) . \quad (40)$$

This is in the lowest order perturbation theory. For higher orders we have terms with products of more interaction potential matrix elements  $\langle |V_I| \rangle$ .

In quantum field theory there is a counterpart to these matrix elements — the *S-matrix*:

$$\langle \beta | V_I | \alpha \rangle + (\text{higher-order terms}) \longrightarrow \langle \beta | S | \alpha \rangle . \quad (41)$$

On one side, *S-matrix* elements can be perturbatively calculated (knowing the interaction Lagrangian/Hamiltonian) with the help of the *Dyson series*

$$S = 1 - i \int d^4 x_1 \mathcal{H}(x_1) + \frac{(-i)^2}{2!} \int d^4 x_1 d^4 x_2 T \{ \mathcal{H}(x_1) \mathcal{H}(x_2) \} + \dots , \quad (42)$$

and on another, we have “golden rules” that associate these matrix elements with cross-sections and decay widths.

It is convenient to express these golden rules in terms of the *Feynman invariant amplitude*  $\mathcal{M}$  which is obtained by stripping some kinematical factors off the  $S$ -matrix:

$$\langle \beta | S | \alpha \rangle = \delta_{\beta\alpha} - i(2\pi)^4 \delta^4(p_\beta - p_\alpha) \mathcal{M}_{\beta\alpha} \prod_{i=\alpha,\beta} \frac{1}{\sqrt{(2\pi)^3 2E_i}}. \quad (43)$$

Now we have two rules:

- Partial decay rate of  $1 \rightarrow 1' + 2' + \dots + n'$

$$d\Gamma = \frac{1}{2E_1} \overline{|\mathcal{M}_{\beta\alpha}|^2} (2\pi)^4 \delta^4(p_1 - p'_1 - \dots - p'_n) \prod_{i=1}^n \frac{d^3 p'_i}{(2\pi)^3 2E'_i}, \quad (44)$$

- Differential cross section for a scattering  $1 + 2 \rightarrow 1' + 2' + \dots + n'$

$$d\sigma = \frac{1}{u_\alpha} \frac{1}{2E_1} \frac{1}{2E_2} \overline{|\mathcal{M}_{\beta\alpha}|^2} (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - \dots - p'_n) \prod_{i=1}^n \frac{d^3 p'_i}{(2\pi)^3 2E'_i}, \quad (45)$$

where  $u_\alpha$  is the relative velocity of particles 1 and 2:

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}, \quad (46)$$

and  $\overline{|\mathcal{M}|^2}$  is the Feynman invariant amplitude averaged over unmeasured particle spins (see Section 6.1). The dimension of  $\mathcal{M}$ , in units of energy, is

- for decays  $[\mathcal{M}] = 3 - n$
- for scattering of two particles  $[\mathcal{M}] = 2 - n$

where  $n$  is the number of produced particles.

So calculation of some observable quantity consists of two stages:

1. Determination of  $\overline{|\mathcal{M}|^2}$ . For this we use the method of Feynman diagrams to be introduced in the next section.
2. Integration over the Lorentz invariant phase space

$$d\text{Lips} = (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - \dots - p'_n) \prod_{i=1}^n \frac{d^3 p'_i}{(2\pi)^3 2E'_i}.$$

## 5 Feynman diagrams

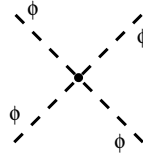
**Example:**  $\phi^4$ -theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4$$

- Free (kinetic) Lagrangian (terms with exactly two fields) determines particles of the theory and their propagators. Here we have just one scalar field:



- Interaction Lagrangian (terms with three or more fields) determines possible vertices. Here, again, there is just one vertex:



We construct all possible diagrams with fixed outer particles. E.g. for scattering of two scalar particles in this theory we would have

$$\mathcal{M}(1 + 2 \rightarrow 3 + 4) = \begin{array}{c} \begin{array}{c} 1 \\ \diagdown \\ \bullet \\ \diagup \\ 2 \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \\ 4 \end{array} \begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \\ \phi \end{array} \begin{array}{c} \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \\ \phi \end{array} \begin{array}{c} \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \end{array} \end{array} + \dots$$

$\xrightarrow{t}$

In these diagrams time flows from left to right. Some people draw Feynman diagrams with time flowing up, which is more in accordance with the way we usually draw space-time in relativity physics.

Since each vertex corresponds to one interaction Lagrangian (Hamiltonian) term in (42), diagrams with loops correspond to higher orders of perturbation theory. Here we will work only to the lowest order, so we will use *tree diagrams* only.

To actually write down the Feynman amplitude  $\mathcal{M}$ , we have a set of *Feynman rules* that associate factors with elements of the Feynman diagram. In particular, to get  $-i\mathcal{M}$  we construct the Feynman rules in the following way:

- the vertex factor is just the  $i$  times the interaction term in the (momentum

space) Lagrangian with all fields removed:

$$i\mathcal{L}_I = -i\frac{g}{4!}\phi^4 \quad \xRightarrow{\text{removing fields}} \quad \begin{array}{c} \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \\ \phi \\ \diagdown \\ \bullet \\ \diagup \\ \phi \end{array} = -i\frac{g}{4!} \quad (47)$$

- the propagator is  $i$  times the inverse of the kinetic operator (defined by the free equation of motion) in the momentum space:

$$\mathcal{L}_{\text{free}} \xrightarrow{\text{Euler-Lagrange eq.}} (\partial_\mu \partial^\mu + m^2)\phi = 0 \quad (\text{Klein-Gordon eq.}) \quad (48)$$

Going to the momentum space using the substitution  $\partial^\mu \rightarrow -ip^\mu$  and then taking the inverse gives:

$$(p^2 - m^2)\phi = 0 \quad \Rightarrow \quad \bullet \text{---} \overset{\phi}{\text{---}} \text{---} \bullet = \frac{i}{p^2 - m^2} \quad (49)$$

(Actually, the correct Feynman propagator is  $i/(p^2 - m^2 + i\epsilon)$ , but for our purposes we can ignore the infinitesimal  $i\epsilon$  term.)

- External lines are represented by the appropriate polarization vector or spinor (the one that stands by the appropriate creation or annihilation operator in the fields (31), (35), (36) and their conjugates):

particle	Feynman rule
ingoing fermion	$u$
outgoing fermion	$\bar{u}$
ingoing antifermion	$\bar{v}$
outgoing antifermion	$v$
ingoing photon	$\epsilon^\mu$
outgoing photon	$\epsilon^{\mu*}$
ingoing scalar	1
outgoing scalar	1

So the tree-level contribution to the scalar-scalar scattering amplitude in this  $\phi^4$  theory would be just

$$-i\mathcal{M} = -i\frac{g}{4!}. \quad (50)$$

■

**Exercise 13** Determine the Feynman rules for the electron propagator and for the only vertex of quantum electrodynamics (QED):

$$\mathcal{L} = \bar{\psi}(i\not{\partial} + e\not{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu . \quad (51)$$

Note that also

$$\begin{array}{c} \mathbf{p} \\ \bullet \longrightarrow \bullet \end{array} = \frac{i \sum_{\sigma} u(\mathbf{p}, \sigma) \bar{u}(\mathbf{p}, \sigma)}{p^2 - m^2} , \quad (52)$$

i.e. the electron propagator is just the scalar propagator multiplied by the polarization sum. It is nice that this generalizes to propagators of all particles. This is very helpful since inverting the photon kinetic operator is non-trivial due to gauge symmetry complications. Hence, propagators of vector particles are

$$\text{massive:} \quad \begin{array}{c} \mathbf{p}, m \\ \bullet \text{---} \text{---} \text{---} \bullet \end{array} = \frac{-i \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2} \right)}{p^2 - m^2} , \quad (53)$$

$$\text{massless:} \quad \begin{array}{c} \mathbf{p} \\ \bullet \text{---} \text{---} \text{---} \bullet \end{array} = \frac{-i g^{\mu\nu}}{p^2} . \quad (54)$$

This is in principle *almost* all we need to know to be able to calculate the Feynman amplitude of any given process. Note that propagators and external-line polarization vectors are determined only by the particle type (its spin and mass) so that the corresponding rules above are not restricted only to the  $\phi^4$  theory and QED, but will apply to all theories of scalars, spin-1 vector bosons and Dirac fermions (such as the standard model). The only additional information we need are the vertex factors.

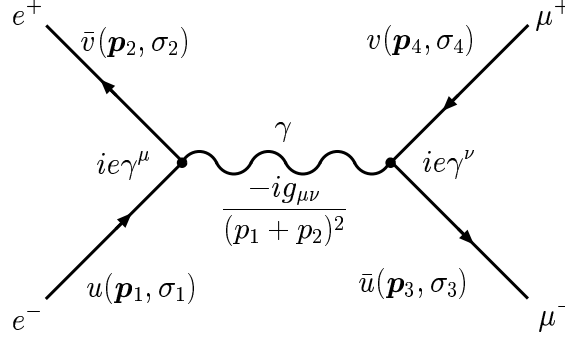
“*Almost*” in the preceding paragraph alludes to the fact that in general Feynman diagram calculation there are several additional subtleties:

- In loop diagrams some internal momenta are undetermined and we have to integrate over those. Also, there is an additional factor (-1) for each closed fermion loop. Since we will do tree-level diagrams only, we can ignore this.
- There are some combinatoric numerical factors when identical fields come into a single vertex.
- Sometimes there is a relative (-) sign between diagrams.
- There is a symmetry factor if there are identical particles in the final state.

These will be explained if we encounter some case where they are relevant.

## 6 $e^+e^- \rightarrow \mu^+\mu^-$ in QED

There is only one contributing tree-level diagram:



We write down the amplitude using the Feynman rules of QED and following fermion lines backwards. Order of lines themselves is unimportant.

$$-i\mathcal{M} = [\bar{u}(\mathbf{p}_3, \sigma_3)(ie\gamma^\nu)v(\mathbf{p}_4, \sigma_4)] \frac{-ig_{\mu\nu}}{(p_1 + p_2)^2} [\bar{v}(\mathbf{p}_2, \sigma_2)(ie\gamma^\mu)u(\mathbf{p}_1, \sigma_1)] , \quad (55)$$

or, introducing abbreviation  $u_1 \equiv u(\mathbf{p}_1, \sigma_1)$ ,

$$\mathcal{M} = \frac{e^2}{(p_1 + p_2)^2} [\bar{u}_3\gamma_\mu v_4][\bar{v}_2\gamma^\mu u_1] . \quad (56)$$

**Exercise 14** Draw Feynman diagram(s) and write down the amplitude for Compton scattering  $\gamma e^- \rightarrow \gamma e^-$ .

### 6.1 Summing over polarizations

If we knew momenta and polarizations of all external particles, we could calculate  $\mathcal{M}$  explicitly. However, experiments are often done with unpolarized particles so we have to sum over the polarizations (spins) of the final particles and average over the polarizations (spins) of the initial ones:

$$|\mathcal{M}|^2 \rightarrow \overline{|\mathcal{M}|^2} = \underbrace{\frac{1}{2} \frac{1}{2} \sum_{\sigma_1 \sigma_2}}_{\text{avg. over initial pol.}} \overbrace{\sum_{\sigma_3 \sigma_4}}^{\text{sum over final pol.}} |\mathcal{M}|^2 . \quad (57)$$

Factors  $1/2$  are due to the fact that each initial fermion has two polarization (spin) states.



(Question: Why we sum probabilities and not amplitudes?)

In the calculation of  $|\mathcal{M}|^2 = \mathcal{M}^* \mathcal{M}$ , the following identity is needed

$$[\bar{u}\gamma^\mu v]^* = [u^\dagger \gamma^0 \gamma^\mu v]^\dagger = v^\dagger \gamma^{\mu\dagger} \gamma^0 u = [\bar{v}\gamma^\mu u]. \quad (58)$$

Thus,

$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4(p_1 + p_2)^4} \sum_{\sigma_{1,2,3,4}} [\bar{v}_4 \gamma_\mu u_3] [\bar{u}_1 \gamma^\mu v_2] [\bar{u}_3 \gamma_\nu v_4] [\bar{v}_2 \gamma^\nu u_1]. \quad (59)$$

## 6.2 Casimir trick

Sums over polarizations are easily performed using the following trick. First we write  $\sum [\bar{u}_1 \gamma^\mu v_2] [\bar{v}_2 \gamma^\nu u_1]$  with explicit spinor indices  $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$ :

$$\sum_{\sigma_1 \sigma_2} \bar{u}_{1\alpha} \gamma_{\alpha\beta}^\mu v_{2\beta} \bar{v}_{2\gamma} \gamma_{\gamma\delta}^\nu u_{1\delta}. \quad (60)$$

We can now move  $u_{1\delta}$  to the front ( $u_{1\delta}$  is just a number, element of  $u_1$  vector, so it commutes with everything), and then use the completeness relations (27) and (28),

$$\begin{aligned} \sum_{\sigma_1} u_{1\delta} \bar{u}_{1\alpha} &= (\not{p}_1 + m_1)_{\delta\alpha}, \\ \sum_{\sigma_2} v_{2\beta} \bar{v}_{2\gamma} &= (\not{p}_2 - m_2)_{\beta\gamma}, \end{aligned}$$

which turn sum (60) into

$$(\not{p}_1 + m_1)_{\delta\alpha} \gamma_{\alpha\beta}^\mu (\not{p}_2 - m_2)_{\beta\gamma} \gamma_{\gamma\delta}^\nu = \text{Tr}[(\not{p}_1 + m_1) \gamma^\mu (\not{p}_2 - m_2) \gamma^\nu]. \quad (61)$$

This means that

$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4(p_1 + p_2)^4} \text{Tr}[(\not{p}_1 + m_1) \gamma^\mu (\not{p}_2 - m_2) \gamma^\nu] \text{Tr}[(\not{p}_4 - m_4) \gamma_\mu (\not{p}_3 + m_3) \gamma_\nu]. \quad (62)$$

Thus we got rid off all the spinors and we are left only with traces of  $\gamma$  matrices. These can be evaluated using the relations from the following section.

## 6.3 Traces and contraction identities of $\gamma$ matrices

All are consequence of the anticommutation relations  $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$ ,  $\{\gamma^\mu, \gamma^5\} = 0$ ,  $(\gamma^5)^2 = 1$ , and of nothing else!

### Trace identities

1. Trace of an odd number of  $\gamma$ 's vanishes:

$$\begin{aligned}
 \text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}) &= \text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}\overbrace{\gamma^5\gamma^5}^1) \\
 \text{(moving } \gamma^5 \text{ over each } \gamma^{\mu_i}) &= -\text{Tr}(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}\gamma^5) \\
 \text{(cyclic property of trace)} &= -\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}\gamma^5\gamma^5) \\
 &= -\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}) \\
 &= 0
 \end{aligned}$$

2.  $\text{Tr } 1 = 4$

- 3.

$$\begin{aligned}
 \text{Tr}\gamma^\mu\gamma^\nu &= \text{Tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) \stackrel{(2.)}{=} 8g^{\mu\nu} - \text{Tr}\gamma^\nu\gamma^\mu = 8g^{\mu\nu} - \text{Tr}\gamma^\mu\gamma^\nu \\
 &\Rightarrow 2\text{Tr}\gamma^\mu\gamma^\nu = 8g^{\mu\nu} \Rightarrow \text{Tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}
 \end{aligned}$$

This also implies:

$$\text{Tr}a\cancel{b} = 4a \cdot b$$

4. **Exercise 15** Calculate  $\text{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ . Hint: Move  $\gamma^\sigma$  all the way to the left, using the anticommutation relations. Then use 3.

*Homework:* Prove that  $\text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n}})$  has  $(2n - 1)!!$  terms.

5.  $\text{Tr}(\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n+1}}) = 0$ . This follows from 1. and from the fact that  $\gamma^5$  consists of even number of  $\gamma$ 's.
6.  $\text{Tr}\gamma^5 = \text{Tr}(\gamma^0\gamma^0\gamma^5) = -\text{Tr}(\gamma^0\gamma^5\gamma^0) = -\text{Tr}\gamma^5 = 0$
7.  $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$ . (Same trick as above, with  $\gamma^\alpha \neq \mu, \nu$  instead of  $\gamma^0$ .)
8.  $\text{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$ , with  $\epsilon^{0123} = 1$ . Careful: convention with  $\epsilon^{0123} = -1$  is also in use.

### Contraction identities

- 1.

$$\gamma^\mu\gamma_\mu = \frac{1}{2}g_{\mu\nu}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}} = g_{\mu\nu}g^{\mu\nu} = 4$$

- 2.

$$\underbrace{\gamma^\mu\gamma^\alpha\gamma_\mu}_{-\gamma_\mu\gamma^\alpha+2g_\mu^\alpha} = -4\gamma^\alpha + 2\gamma^\alpha = -2\gamma^\alpha$$

3. **Exercise 16** Contract  $\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu$ .

4.  $\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\mu = -2\gamma^\gamma \gamma^\beta \gamma^\alpha$

**Exercise 17** Calculate traces in  $\overline{|\mathcal{M}|^2}$ :

$$\text{Tr}[(\not{p}_1 + m_1)\gamma^\mu(\not{p}_2 - m_2)\gamma^\nu] = ?$$

$$\text{Tr}[(\not{p}_4 - m_4)\gamma_\mu(\not{p}_3 + m_3)\gamma_\nu] = ?$$

**Exercise 18** Calculate  $\overline{|\mathcal{M}|^2}$

## 6.4 Kinematics in the center-of-mass frame

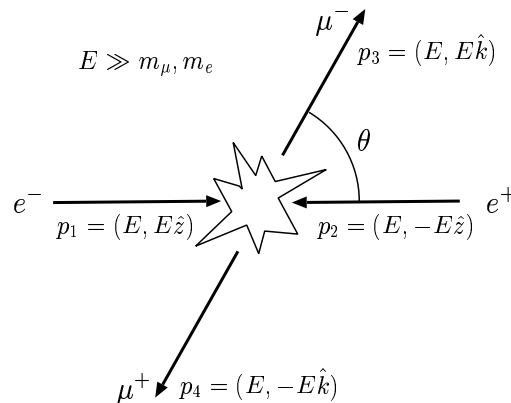
In  $e^+e^-$  colliders often  $p_i \gg m_e, m_\mu, i = 1, \dots, 4$ , so we can take

$$m_i \rightarrow 0 \quad \text{“high-energy” or “extreme relativistic” limit}$$

Then

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{(p_1 + p_2)^4} [(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] \quad (63)$$

To calculate scattering cross-section  $\sigma$  we have to specialize to some particular frame ( $\sigma$  is *not* frame-independent). For  $e^+e^-$  colliders the most relevant is the center-of-mass (CM) frame:



**Exercise 19** Express  $\overline{|\mathcal{M}|^2}$  in terms of  $E$  and  $\theta$ .

## 6.5 Integration over two-particle phase space

Now we can use the “golden rule” (45) for the  $1+2 \rightarrow 3+4$  differential scattering cross-section

$$d\sigma = \frac{1}{u_\alpha} \frac{1}{2E_1} \frac{1}{2E_2} |\overline{\mathcal{M}}|^2 d\text{Lips}_2 \quad (64)$$

where two-particle phase space to be integrated over is

$$d\text{Lips}_2 = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}. \quad (65)$$

First we integrate over four out of six integration variables, and we do this in general frame.  $\delta$ -function makes the integration over  $d^3p_4$  trivial giving

$$d\text{Lips}_2 = \frac{1}{(2\pi)^2 4E_3E_4} \delta(E_1 + E_2 - E_3 - E_4) \underbrace{d^3p_3}_{\mathbf{p}_3^2 d|\mathbf{p}_3| d\Omega_3} \quad (66)$$

Now we integrate over  $d|\mathbf{p}_3|$  by noting that  $E_3$  and  $E_4$  are functions of  $|\mathbf{p}_3|$

$$\begin{aligned} E_3 &= E_3(|\mathbf{p}_3|) = \sqrt{\mathbf{p}_3^2 + m_3^2}, \\ E_4 &= \sqrt{\mathbf{p}_4^2 + m_4^2} = \sqrt{\mathbf{p}_3^2 + m_4^2}, \end{aligned}$$

and by  $\delta$ -function relation

$$\delta(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_3^2 + m_4^2}) = \delta[f(|\mathbf{p}_3|)] = \frac{\delta(|\mathbf{p}_3| - |\mathbf{p}_3^{(0)}|)}{|f'(|\mathbf{p}_3|)|_{|\mathbf{p}_3|=|\mathbf{p}_3^{(0)}|}}. \quad (67)$$

Here  $|\mathbf{p}_3|$  is just the integration variable and  $|\mathbf{p}_3^{(0)}|$  is the zero of  $f(|\mathbf{p}_3|)$  i.e. the *actual* momentum of the third particle. After we integrate over  $d|\mathbf{p}_3|$  we put  $|\mathbf{p}_3^{(0)}| \rightarrow |\mathbf{p}_3|$ .

Since

$$f'(|\mathbf{p}_3|) = -\frac{E_3 + E_4}{E_3E_4} |\mathbf{p}_3|, \quad (68)$$

we get

$$d\text{Lips}_2 = \frac{|\mathbf{p}_3| d\Omega}{16\pi^2 (E_1 + E_2)}. \quad (69)$$

Now we again specialize to the CM frame and note that the flux factor is

$$4E_1E_2u_\alpha = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2m_2^2} = 4|\mathbf{p}_1|(E_1 + E_2), \quad (70)$$

giving finally

$$\frac{d\sigma_{\text{CM}}}{d\Omega} = \frac{1}{64\pi^2(E_1 + E_2)^2} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} |\overline{\mathcal{M}}|^2. \quad (71)$$

Note that we kept masses in each step so this formula is generally valid for any CM scattering.

For our particular  $e^-e^+ \rightarrow \mu^-\mu^+$  scattering this gives the final result for differential cross-section (introducing the fine structure constant  $\alpha = e^2/(4\pi)$ )

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta). \quad (72)$$

**Exercise 20** Integrate this to get the total cross section  $\sigma$ .

Note that it is obvious that  $\sigma \propto \alpha^2$ , and that dimensional analysis requires  $\sigma \propto 1/E^2$ , so only angular dependence  $(1 + \cos^2 \theta)$  tests QED as a theory of leptons and photons.

## 6.6 Summary of steps

To recapitulate, calculating scattering cross-section (or decay width) consists of the following steps:

1. drawing the Feynman diagram(s)
2. writing  $-i\mathcal{M}$  using the Feynman rules
3. squaring  $\mathcal{M}$  and using the Casimir trick to get traces
4. evaluating traces
5. applying kinematics of the chosen frame
6. integrating over the phase space

## 6.7 Mandelstam variables

Mandelstam variables  $s$ ,  $t$  and  $u$  are often used in scattering calculations. They are defined (for  $1 + 2 \rightarrow 3 + 4$  scattering) as

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

**Exercise 21** Prove that  $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

This means that only two Mandelstam variables are independent. Their main advantage is that they are Lorentz invariant which renders them convenient for Feynman amplitude calculations. Only at the end we can exchange them for “experimenter’s” variables  $E$  and  $\theta$ .

**Exercise 22** Express  $\overline{|\mathcal{M}|^2}$  for  $e^-e^+ \rightarrow \mu^-\mu^+$  scattering in terms of Mandelstam variables.

## Appendix: Doing Feynman diagrams on a computer

There are several computer programs that can perform some or all of the steps in the calculation of Feynman diagrams. Here is one simple session with one of such programs, `FeynCalc` for Wolfram's *Mathematica*, where we calculate the same process,  $e^-e^+ \rightarrow \mu^-\mu^+$ , that we just calculated by hand.

*FeynCalcDemo.nb*

1

### FeynCalc demonstration

This *Mathematica* notebook demonstrates computer calculation of Feynman invariant amplitude for  $e^-e^+ \rightarrow \mu^-\mu^+$  scattering, using FeynCalc package.

First we load FeynCalc into Mathematica

```
In[1]:= << HighEnergyPhysics`fc`
```

**FeynCalc** 4.1.0.3b Evaluate ?FeynCalc for help or visit [www.feyncalc.org](http://www.feyncalc.org)

Spin-averaged Feynman amplitude squared  $\overline{|M|^2}$  after using Feynman rules and applying the Casimir trick:

$$\text{In}[2] := \text{Msq} = \frac{e^4}{4(p_1 + p_2)^4} \frac{\text{Contract}[\text{Tr}[(\text{GS}[p_1] + m_e) \cdot \text{GA}[\mu] \cdot (\text{GS}[p_2] - m_e) \cdot \text{GA}[\nu]] \cdot \text{Tr}[(\text{GS}[p_4] - m_\mu) \cdot \text{GA}[\mu] \cdot (\text{GS}[p_3] + m_\mu) \cdot \text{GA}[\nu]]]]}{4(p_1 + p_2)^4}$$

$$\text{Out}[2] = \frac{1}{4(p_1 + p_2)^4} (e^4 (64 \text{mm}^2 m_e^2 + 32 p_3 \cdot p_4 m_e^2 + 32 \text{mm}^2 p_1 \cdot p_2 + 32 p_1 \cdot p_4 p_2 \cdot p_3 + 32 p_1 \cdot p_3 p_2 \cdot p_4))$$

Traces were evaluated and contractions performed automatically. Now we introduce Mandelstam variables by substitution rules,

```
In[3]:= prod[a_, b_] := Pair[Momentum[a], Momentum[b]];
mandelstam = {prod[p1, p2] -> (s - me^2 - me^2) / 2, prod[p3, p4] -> (s - mm^2 - mm^2) / 2,
prod[p1, p3] -> (t - me^2 - mm^2) / 2, prod[p2, p4] -> (t - me^2 - mm^2) / 2,
prod[p1, p4] -> (u - me^2 - mm^2) / 2, prod[p2, p3] -> (u - me^2 - mm^2) / 2, (p1 + p2) -> sqrt[s]};
```

and apply these substitutions to our amplitude:

```
In[5]:= Msq /. mandelstam
```

$$\text{Out}[5] = \frac{1}{4s^2} (e^4 (64 \text{mm}^2 m_e^2 + 16(s - 2 \text{mm}^2) m_e^2 + 8(-m_e^2 - \text{mm}^2 + t)^2 + 8(-m_e^2 - \text{mm}^2 + u)^2 + 16 \text{mm}^2 (s - 2 m_e^2)))$$

This result can be simplified by eliminating one Mandelstam variable:

```
In[6]:= Simplify[TrickMandelstam[%, s, t, u, 2 me^2 + 2 mm^2]]
```

$$\text{Out}[6] = \frac{2 e^4 (2 m_e^4 + 4 (\text{mm}^2 - u) m_e^2 + 2 \text{mm}^4 + s^2 + 2 u^2 - 4 \text{mm}^2 u + 2 s u)}{s^2}$$

If we go to ultra-relativistic limit, we get result in agreement with our hand calculation:

```
In[7]:= Simplify[% /. {mm -> 0, me -> 0}]
```

$$\text{Out}[7] = \frac{2 e^4 (t^2 + u^2)}{s^2}$$